A Complex Transposition Theorem with Applications to Complex Programming

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INTRODUCTION

The Motzkin transposition theorem [2] states that the system

$$Ax > 0, \qquad Bx \geqslant 0, \qquad Cx = 0 \tag{1}$$

has a solution if and only if

$$A^t y^1 + B^t y^2 + C^t y^3 = 0, \qquad y^1 \geqslant 0, \quad y^1 \neq 0, \quad y^2 \geqslant 0, \tag{2}$$

has none. The theorem includes the case in which B or C or both (but not A) is missing. Here A, B, and C are real matrices; x, y^1, y^2, and y^3 are real column vectors. Superscript t denotes transpose.

In this note, we give a complex version of the transposition theorem. Our result is then applied to provide a simple proof of the duality theorem for linear programming in complex space recently given by Levinson [1].

RESULTS

Henceforth, A, B, and C will denote complex matrices. z, w, w^1, w^2, w^3, w^4, w^5, b, and c will denote complex column vectors. ξ will denote

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a complex scalar. The entries of A, B, C, b, and c are constants, whereas those of z, w, w^1, w^2, w^3, w^4, w^5, and ξ are variables. α and β are real constant vectors with $0 \leqslant \alpha \leqslant \pi/2$, $0 \leqslant \beta \leqslant \pi/2$. Here, and throughout, $\pi/2$ denotes the vector of appropriate dimension with $\pi/2$ in each entry. Arguments of complex numbers are restricted to the interval $(-\pi, \pi]$. The complex number zero will be assigned the argument zero. The asterisk denotes conjugate transpose. It is assumed that vectors and matrices are of appropriate dimension so as to make indicated operations meaningful.

THEOREM 1. *The system*

$$\operatorname{Re} Az > 0, \qquad |\arg Bz| \leqslant \pi/2 - \alpha, \qquad Cz = 0 \tag{3}$$

has a solution if and only if

$$\begin{gathered} A^*w^1 + B^*w^2 + C^*w^3 = 0, \qquad \operatorname{Im} w^1 = 0, \quad \operatorname{Re} w^1 \geqslant 0, \quad w^1 \neq 0, \\ |\arg w^2| \leqslant \alpha \end{gathered} \tag{4}$$

has none. B or C, or both, but not A, may be missing.

Proof. For simplicity of notation, in this proof the subscript R will be used to denote real part and the subscript I to denote imaginary part, e.g., $A = A_R + iA_I$. All other subscripts will denote vector components. M^1, M^2, and M^3 will denote, respectively, the set (possibly empty) of integers j such that $\alpha_j = 0$; $0 < \alpha_j < \pi/2$; and $\alpha_j = \pi/2$.

Equation (4) is easily seen to be equivalent to the following system of linear equations and inequalities in real space:

$$\begin{gathered} A_R^t w_R^1 + A_I^t w_I^1 + B_R^t w_R^2 + B_I^t w_I^2 + C_R^t w_R^3 + C_I^t w_I^3 = 0. \\ -A_I^t w_R^1 + A_R^t w_I^1 - B_I^t w_R^2 + B_R^t w_I^2 - C_I^t w_R^3 + C_R^t w_I^3 = 0. \\ w_R^1 \geqslant 0, \qquad w_R^1 \neq 0, \qquad w_I^1 = 0. \\ w^2_{R_j} \geqslant 0, \qquad w^2_{I_j} = 0, \qquad j \in M^1. \\ w^2_{R_j} \tan \alpha_j + w^2_{I_j} - t_j^1 = 0, \qquad j \in M^2, \\ w^2_{R_j} \tan \alpha_j - w^2_{I_j} - t_j^2 = 0, \qquad j \in M^2, \\ t_j^1 \geqslant 0, \qquad t_j^2 \geqslant 0, \qquad j \in M^2. \end{gathered} \tag{5}$$

$$w^2_{R_j} \geqslant 0, \qquad j \in M^3.$$

t^1 and t^2 are appropriate real slack vectors introduced to make the first two lines of (5) equalities.

We now have a system in real space corresponding to (2). By the Motzkin transposition theorem, a solution exists if, and only if, the following solution in real space has none:

$$A_R x^1 - A_I x^2 > 0.$$

$$A_I x^1 + A_R x^2 + x^3 = 0.$$

$$(B_R x^1 - B_I x^2)_j \geqslant 0, \qquad j \in M^1,$$

$$(B_I x^1 + B_R x^2)_j + x_j{}^4 = 0, \qquad j \in M^1.$$

$$(B_R x^1 - B_I x^2)_j + x_j{}^5 \tan \alpha_j + x_j{}^6 \tan \alpha_j = 0, \qquad j \in M^2,$$

$$(B_I x^1 + B_R x^2)_j + x_j{}^5 - x_j{}^6 = 0, \qquad j \in M^2,$$

$$- x_j{}^5 \geqslant 0, \qquad - x_j{}^6 \geqslant 0, \qquad j \in M^2.$$

$$(B_R x^1 - B_I x^2)_j \geqslant 0, \qquad j \in M^3,$$

$$(B_I x^1 - B_R x^2)_j = 0, \qquad j \in M^3.$$

$$C_R x^1 - C_I x^2 = 0,$$

$$C_I x^1 + C_R x^2 = 0.$$

Letting $z = x^1 + ix^2$, i.e., $z_R = x^1$ and $z_I = x^2$, the above system of equations and inequalities can be rewritten in the following way:

$$(Az)_R > 0.$$

$$(Az)_I \text{ unrestricted.}$$

$$\begin{aligned} &(Bz)_{R_j} \geqslant 0, \qquad j \in M^1, \\ &(Bz)_{I_j} \text{ unrestricted}, \qquad j \in M^1. \end{aligned} \tag{6}$$

$$\begin{aligned} (Bz)_{R_j} &= (-x_j^5 - x_j^6)\tan\alpha_j, & j &\in M_2, \\ (Bz)_{I_j} &= -x_j^5 + x_j^6, & j &\in M_2, \\ -x_j^5 \geqslant 0, \qquad -x_j^6 &\geqslant 0, & j &\in M_2. \end{aligned} \tag{7}$$

$$\begin{aligned} (Bz)_{R_j} &\geqslant 0, & j &\in M^3, \\ (Bz)_{I_j} &= 0, & j &\in M^3. \end{aligned} \tag{8}$$

$$Cz = 0.$$

The proof will be complete if we can show that (6), (7), and (8) are equivalent to $|\arg Bz| \leqslant \pi/2 - \alpha$.

Since $\alpha_j = 0$, $j \in M^1$, (6) is equivalent to $|\arg(Bz)_j| \leqslant \pi/2 = \pi/2 - \alpha_j$, $j \in M^1$. Since $\alpha_j = \pi/2$, $j \in M^3$, (8) is equivalent to

$$|\arg(Bz)_j| \leqslant 0 = \pi/2 - \alpha_j, \qquad j \in M^3.$$

Since $0 < \alpha_j < \pi/2$, $j \in M^2$, (7) implies

$$\frac{(Bz)_{R_j}}{|(Bz)_{I_j}|} = \frac{(-x_j^5 - x_j^6)\tan\alpha_j}{|-x_j^5 + x_j^6|} \geqslant \tan\alpha_j = \frac{1}{\tan(\pi/2 - \alpha_j)}, \qquad j \in M^2;$$

or

$$|(Bz)_{I_j}|/(Bz)_{R_j} \leqslant \tan(\pi/2 - \alpha_j), \qquad j \in M^2.$$

Hence, (7) implies

$$|\arg(Bz)_j| \leqslant \pi/2 - \alpha_j, \qquad j \in M_2. \tag{9}$$

Inequality (9) implies (7), in the sense that if there exists a z satisfying (9), then there exist x^5 and x^6 (defined by

$$x_j^5 = \{[-(Bz)_{R_j}/\tan\alpha_j] - (Bz)_{I_j}\}/2,$$

$$x_j^6 = \{[-(Bz)_{R_j}/\tan\alpha_j] + (Bz)_{I_j}\}/2\ )$$

that satisfy (7).

The next theorem was established by Levinson [1, Theorem 2.2]. We offer another proof of Levinson's result.

THEOREM 2. *If the complex linear programming problem* (P)

$$\min \operatorname{Re} c^* z$$

subject to

$$|\arg(Az - b)| \leqslant \pi/2 - \beta$$

and

$$|\arg z| \leqslant \alpha$$

has an optimal solution, then the dual problem

$$\max \operatorname{Re} b^* w$$

subject to

$$|\arg(-A^* w + c)| \leqslant \pi/2 - \alpha$$

and

$$|\arg w| \leqslant \beta$$

has an optimal solution; and, for optimal z^0 *and* w^0,

$$\operatorname{Re} b^* w^0 = \operatorname{Re} c^* z^0.$$

Proof. Let $\delta \equiv \operatorname{Re} c^* z^0$. Solvability of (P) implies solvability of the system

$$|\arg(Az - b\xi)| \leqslant \pi/2 - \beta,$$

$$|\arg z| \leqslant \alpha,$$

$$|\arg(-c^* z + \delta\xi)| \leqslant \pi/2,$$

$$|\arg \xi| \leqslant 0,$$

$$\operatorname{Re} \xi > 0,$$

which, by Theorem 1, implies insolvability of the system

$$A^* w^1 + w^2 - c w^3 = 0,$$

$$-b^* w^1 + \delta w^3 + w^4 + w^5 = 0,$$

$$|\arg w^1| \leqslant \beta,$$

$$|\arg w^2| \leqslant \pi/2 - \beta,$$

$$|\arg w^3| \leqslant 0,$$

$$|\arg w^4| \leqslant \pi/2,$$

$$\operatorname{Re} w^5 > 0, \qquad \operatorname{Im} w^5 = 0.$$

Eliminating w^2 and w^4 yields

$$|\arg(- A^*w^1 + cw^3)| \leqslant \pi/2 - \alpha$$

and

$$|\arg(b^*w^1 - \delta w^3 - w^5)| \leqslant \pi/2,$$

or, equivalently,

$$\operatorname{Re}(b^*w^1 - \delta w^3 - w^5) \geqslant 0.$$

Eliminating w^5 gives

$$\operatorname{Re} b^*w^1 > \operatorname{Re} \delta w^3.$$

Thus, solvability of (P) implies insolvability of

$$|\arg(- A^*w^1 + cw^3)| \leqslant \pi/2 - \alpha,$$

$$|\arg w^1| \leqslant \beta,$$

$$\operatorname{Re} b^*w^1 > \operatorname{Re} \delta w^3,$$

$$|\arg w^3| \leqslant 0,$$

and hence also the insolvability of

$$|\arg(- A^*w^1 + c)| \leqslant \pi/2 - \alpha,$$

$$|\arg w^1| \leqslant \beta,$$

$$\operatorname{Re} b^*w^1 > \delta.$$

Now, z^0 optimal for (P) implies insolvability of the system

$$|\arg(Az - b\xi)| \leqslant \pi/2 - \beta,$$
$$|\arg z| \leqslant \alpha,$$
$$\mathrm{Re}(-c^*z + \delta\xi) > 0,$$
$$|\arg \xi| \leqslant 0,$$
$$\mathrm{Re}\,\xi > 0,$$

which, by Theorem 1, implies solvability of the system

$$A^*w^1 + w^2 - cw^3 = 0,$$
$$-b^*w^1 + \delta w^3 + w^4 + w^5 = 0,$$
$$|\arg w^1| \leqslant \beta,$$
$$|\arg w^2| \leqslant \pi/2 - \alpha,$$
$$|\arg w^4| \leqslant \pi/2,$$
$$\mathrm{Re}(w^3, w^5) \geqslant 0, \qquad (w^3, w^5) \neq 0, \qquad \mathrm{Im}(w^3, w^5) = 0.$$

Eliminating w^2 and w^4 yields

$$|\arg(-A^*w_1 + cw^3)| \leqslant \pi/2 - \alpha$$

and

$$|\arg(b^*w_1 - \delta w^3 - w^5)| \leqslant \pi/2,$$

or, equivalently,

$$\mathrm{Re}(b^*w_1 - \delta w^3 - w^5) \geqslant 0.$$

Now, $\mathrm{Re}\,w^5 > 0$ leads to a contradiction. Hence $\mathrm{Re}\,w^5 = 0$ and $\mathrm{Re}\,w^3 > 0$, from which follows the solvability of

$$|\arg(-A^*w^1 + c)| \leqslant \pi/2 - \alpha,$$
$$|\arg w^1| \leqslant \beta,$$
$$\mathrm{Re}\,b^*w^1 \geqslant \delta$$

and the theorem is proved. (Compare the method of proof of Theorem 2 with that of Motzkin [3].)

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